

Supersymmetric probability distributions

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Abstract

We use anticommuting variables to study probability distributions of random variables, that are solutions of Langevin’s equation. We show that the probability density always enjoys “worldpoint supersymmetry”. The partition function, however, may not. We find that the domain of integration can acquire a boundary, that implies that the auxiliary field has a non-zero expectation value, signalling spontaneous supersymmetry breaking. This is due to the presence of “fermionic” zeromodes, whose contribution cannot be cancelled by a surface term. This we prove by an explicit calculation of the regularized partition function, as well as by computing the moments of the auxiliary field and checking whether they satisfy the identities implied by Wick’s theorem. Nevertheless, supersymmetry manifests itself in the identities that are satisfied by the moments of the scalar, whose expressions we can calculate, for all values of the coupling constant. We also provide some quantitative estimates concerning the visibility of supersymmetry breaking effects in the identities for the moments and remark that the shape of the distribution of the auxiliary field can influence quite strongly how easy it would be to mask them, since the expectation value of the auxiliary field doesn’t coincide with its typical value.

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1 Introduction

Let us consider a stochastic process, of commuting random variables, $x(\tau)$, that satisfies the Langevin equation

$$\frac{dx(\tau)}{d\tau} = -\frac{\partial U(x(\tau))}{\partial x(\tau)} + \eta(\tau) \quad (1)$$

Here $\eta(\tau)$ is a Gaussian stochastic process, i.e. its correlation functions satisfy the relations

$$\begin{aligned} \langle \eta(\tau) \rangle &= 0 \\ \langle \eta(\tau_1)\eta(\tau_2) \rangle &= \delta(\tau_1 - \tau_2) \\ \langle \eta(\tau_1)\eta(\tau_2) \cdots \eta(\tau_{2n}) \rangle &= \sum_{\pi} \langle \eta_{\pi(1)}\eta_{\pi(2)} \rangle \langle \eta_{\pi(3)}\eta_{\pi(4)} \rangle \cdots \langle \eta_{\pi(2n-1)}\eta_{\pi(2n)} \rangle \end{aligned} \quad (2)$$

where the sum is over all permutations, π , of the index values $1, 2, 3, \dots, 2n$.

We shall assume that $U(x(\tau))$ is an ultra-local functional of $x(\tau)$, i.e. doesn't contain any derivatives with respect to τ and discuss in our conclusions what happens when $U(x(\tau))$ is, simply, a local functional of $x(\tau)$.

We are interested in computing the probability distribution of the limiting value, $x_{\infty} \equiv \lim_{\tau \rightarrow \infty} x(\tau)$. In this limit, the Gaussian stochastic process, $\eta(\tau)$, is a Gaussian variable $\eta_{\infty} \equiv \eta$ and the Langevin equation takes the form (to simplify notation we set $x_{\infty} \equiv x$ and $\eta_{\infty} \equiv \eta$)

$$\eta = \frac{dU(x)}{dx} \quad (3)$$

This relation indicates that $dU(x)/dx$ is drawn from a Gaussian distribution. We are interested in the distribution, $\rho(x)$, of x and will try to determine it from its moments, $\langle x^l \rangle$. The partition function is given by the expression

$$Z = \int_{-\infty}^{\infty} dx d\eta e^{-\frac{\eta^2}{2}} \delta\left(\eta - \frac{dU}{dx}\right) = \int_{-\infty}^{\infty} dx \left| \frac{d^2U}{dx^2} \right| e^{-\frac{1}{2}\left(\frac{dU}{dx}\right)^2} \quad (4)$$

This holds only if $U''(x) \neq 0$ for all values of x . If $U''(x)$ can vanish then the second equality does not hold—a well known fact of freshman calculus: we must partition the domain of integration into intervals, where the Jacobian is of fixed sign. The point(s) where the Jacobian vanishes are “excised” and the integrals computed by limiting procedure. We would like to explore in greater detail the symmetries of the generating function Z , written as an integral over x , when this limiting procedure is necessary. The reason is that this toy model will be useful as a laboratory for field theories. In fact it describes the boundary degree(s) of freedom of supersymmetric quantum mechanics in one space dimension.

The plan of our paper is the following:

In section 2 we show that the “classical” effective action, obtained by exponentiating the Jacobian, is invariant under a transformation, linear in x , whose parameter is an anticommuting

variable, thereby displaying worldpoint supersymmetry. Since we must introduce a regularization procedure, in order to integrate over the fermionic zeromodes, it is not obvious that this symmetry is a symmetry of the partition function, as well. We present an argument, under what circumstances the regularized partition function, once the cutoff is removed, will, indeed, respect supersymmetry.

To gain further insight, we study the infinite set of relations between the moments, the zero-dimensional avatars of the Ward–Takahashi identities of field theory.

This we do in section 3. We first establish the identities that are satisfied by the moments of the auxiliary field. These, however, depend quite loosely on the particular dynamical model. The identities between the moments of the “physical variable” are obtained by replacing the auxiliary field in these identities by its equation of motion. We must, therefore, show that the moments of the physical variable exist, for these identities to be well-defined.

We compute these moments in two, different, ways: on the one hand we invert the “Nicolai map”, $F = dU/dx$ —which can be done exactly, for the example we consider—on the other hand we compute the moments of the scalar variable from the moments of $\exp(-(dU/dx)^2/2)$.

In section 4 we insert these moments in the identities satisfied by the auxiliary field and deduce relations for the moments of the physical variable itself. These can be evaluated either by using properties of special functions, or numerically.

In section 5 we discuss the challenges that the generalization to finite dimensions poses and how they might be tackled, as well as the issues for describing “target space” supersymmetry in this formalism.

2 Worldpoint Supersymmetry

We would like to write

$$\left| \frac{d^2 U(x)}{dx^2} \right| e^{-\frac{1}{2} \left(\frac{dU}{dx} \right)^2} \equiv e^{-S_{\text{eff}}} \quad (5)$$

One way, of course, would be to write

$$\left| \frac{d^2 U(x)}{dx^2} \right| = e^{\log \left| \frac{d^2 U(x)}{dx^2} \right|} = e^{\frac{1}{2} \log \left(\frac{d^2 U(x)}{dx^2} \right)^2}$$

In field theory this expression would become $\exp(\text{Tr} \log U''(x))$ and, since $U(x)$ in field theory is a local functional of the fields, this contribution to the action would be non-local.

On the other hand, as is well known, we may introduce it in a local way (in field theory) if we use two anticommuting variables, ψ_α , $\alpha = 1, 2$:

$$\left| \frac{d^2 U(x)}{dx^2} \right| = \int d\psi_1 d\psi_2 e^{\frac{1}{2} \psi_\alpha \varepsilon^{\alpha\beta} U''(x) \psi_\beta} \quad (6)$$

with

$$\varepsilon^{\alpha\beta} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The anticommuting variables introduced here are *not* ghosts, but as “physical” as x . There isn’t any notion of spin so the spin–statistics theorem is vacuous. (The only way they could be ghosts would be if $U''(x)$ were imaginary.)

The effective action, therefore, becomes

$$S_{\text{eff}}(x, \psi) = \frac{1}{2} \left(\frac{dU}{dx} \right)^2 - \frac{1}{2} \psi_\alpha \varepsilon^{\alpha\beta} U''(x) \psi_\beta \quad (7)$$

We notice a mismatch: one commuting degree of freedom versus two anticommuting degrees of freedom. We may restore equality by introducing another commuting variable that we will call, for historical reasons, F [2]:

$$e^{-\frac{1}{2} \left(\frac{dU}{dx} \right)^2} = \int_{-\infty}^{\infty} dF e^{-\frac{1}{2} F^2 + iF \frac{dU}{dx}} = i \int_{-\infty}^{\infty} dF e^{\frac{1}{2} F^2 - F \frac{dU}{dx}} \quad (8)$$

Here we remark that the variable F , in the field theory generalization, has an ultra–local propagator and that, were this propagator local, F would be a ghost. The final expression for the classical effective action, therefore, is

$$S_{\text{eff}}(x, \psi, F) = -\frac{1}{2} F^2 + F \frac{dU}{dx} - \frac{1}{2} \psi_\alpha \varepsilon^{\alpha\beta} \frac{d^2 U}{dx^2} \psi_\beta \quad (9)$$

This action is invariant under the following transformation[2]:

$$\begin{aligned} \delta x &= \zeta_\alpha \varepsilon^{\alpha\beta} \psi_\beta \\ \delta \psi_\alpha &= \zeta_\alpha F \\ \delta F &= 0 \end{aligned} \quad (10)$$

The parameter(s), ζ_α , $\alpha = 1, 2$, are anticommuting variables so we remark that this transformation mixes commuting and anticommuting variables. It is thus called a supersymmetry transformation. Since there is one anticommuting pair of variables, ζ_α , this is $\mathcal{N} = 1$ supersymmetry. It doesn’t seem to depend on the explicit form of $U(x)$, which is the zero–dimensional avatar of the superpotential (this is due to the fact that we have introduced the auxiliary field, F , which leads to a linear realization [3, 2]). When $\langle F \rangle \neq 0$, these relations imply that the fermion becomes the zero–dimensional avatar of the goldstino.

If we compute the anticommutator, generated by the transformations (10), we find that

$$[\zeta Q, \eta Q] = 0 \Leftrightarrow \{Q_\alpha, Q_\beta\} = 0 \quad (11)$$

This relation implies that eigenstates of these operators with nonzero eigenvalue are paired. It is silent about the existence, or not of eigenstates of zero eigenvalue. If such latter states exist, then supersymmetry is realized—if not, it is broken. The resolution depends on the dynamics.

Since it is the dynamics that will interest us, we shall work with the partition function and the identities of its moments and not consider the algebra itself in what follows. The calculations are much more direct and we need make much fewer assumptions.

The “equation of motion” for F , obtained by varying the effective action, is

$$F = \frac{dU}{dx} \quad (12)$$

From eq. (3) we deduce that $F = \eta$, which means that the “auxiliary variable”, F , which was introduced to render the supersymmetry transformations linear in x , is drawn from the same distribution as the noise.

If $U(x)$ is a quadratic function of x , then the relation between F and x is linear. This is the case of the free theory (provided the relation is invertible). If $U(x)$ is not quadratic, the standard approach [2, 6, 7] is to express the “physical” field, x , in terms of the “auxiliary” field, F through a perturbation expansion about a reference configuration. Supersymmetry implies identities between the correlation functions that follow from Wick’s theorem applied to the Gaussian distribution of the auxiliary field.

Here we would like to understand what happens to supersymmetry, in this formalism, if $\varepsilon^{\alpha\beta}U''(x)$ has zeromodes.

In the next section we shall write down the stochastic identities for the case of the cubic superpotential, without recourse to any perturbative expansion.

We shall use two, logically independent, methods: The first consists in solving the equation of motion for the auxiliary field and expressing the moments of the scalar in terms of the moments of the auxiliary field.

In the second approach we shall compute the moments of the scalar from the moments of the classical action of the scalar. We end up with identical expressions in both cases—since we solved the equation of motion of the auxiliary field exactly. This is just a test case. In more general situations we cannot solve the equation for the auxiliary field and the second approach is more effective.

We conclude with a discussion of directions for further inquiry.

3 Worldpoint Supersymmetry beyond the classical action

An example that illustrates these issues is that of the zero-dimensional model with cubic superpotential:

$$S = -\frac{1}{2}F^2 + F \left(m^2x + \frac{\lambda}{2}x^2 \right) - \frac{1}{2}\psi_\alpha \varepsilon^{\alpha\beta} (m^2 + \lambda x) \psi_\beta \quad (13)$$

When we integrate out the “fermions” and the auxiliary variable, F , we find

$$Z = \int_{-\infty}^{\infty} dx \underbrace{\text{sign}(m^2 + \lambda x) (m^2 + \lambda x)}_{|m^2 + \lambda x|} e^{-\frac{1}{2}(m^2x + \frac{\lambda}{2}x^2)^2}$$

(In fact here there's a sign issue that doesn't have anything to do with the sign of the Jacobian, namely, when we integrate over two anticommuting variables, ψ_1 and ψ_2

$$\int d\psi_1 d\psi_2 e^{A\psi_1\psi_2} = \int d\psi_1 d\psi_2 (1 + A\psi_1\psi_2) = -A$$

This is a global sign and doesn't play any role in the calculation of correlation functions—we fix it once and for all.)

We remark that $U''(x) = m^2 + \lambda x$ vanishes at $x = -m^2/\lambda$. If we try to compute $\langle x^p \rangle$ by direct sampling:

$$\langle x^p \rangle = \frac{\int dx x^p \text{sign}(m^2 + \lambda x) (m^2 + \lambda x) e^{-\frac{1}{2}(m^2 x + \frac{\lambda}{2} x^2)^2}}{\int dx \text{sign}(m^2 + \lambda x) (m^2 + \lambda x) e^{-\frac{1}{2}(m^2 x + \frac{\lambda}{2} x^2)^2}} \quad (14)$$

we see that this expression isn't well-defined, since the naïve action

$$S_{\text{naive}} = \frac{1}{2} \left(m^2 x + \frac{\lambda}{2} x^2 \right)^2 - \ln |m^2 + \lambda x| \quad (15)$$

is not bounded from below: $\ln |m^2 + \lambda x| \rightarrow -\infty$ as $x \rightarrow -m^2/\lambda$. We remark, further, that the “classical action”,

$$S \equiv \frac{1}{2} \left(m^2 x + \frac{\lambda}{2} x^2 \right)^2$$

has a double well structure, with degenerate minima at $x = 0$ and $x = -2m^2/\lambda$ and a maximum at $x = -m^2/\lambda$. At this maximum the (Majorana) “mass” of the “fermion” vanishes—“chiral symmetry” might be realized. At the minima “chiral symmetry” is “broken”—the “fermion” is “massive” and its mass is equal to that of the “scalar”—supersymmetry is unbroken. If we study the system in perturbation theory, we can only access the vicinity of the minima and we should find unbroken “supersymmetry”: $m_F = U''(x^*)$ and $m_B^2 = (U''(x^*)^2 + U'(x^*)U'''(x^*)) = m_F^2$, since x^* is root of $U'(x^*) = 0$; broken “chiral symmetry” ($m_F \neq 0$) and no hint of the instability at the maximum. (If we draw e^{-S} , we notice that it has support on the whole real axis.) When we try to compute the moments beyond perturbation theory, however, we will generate configurations around the maximum—where the Jacobian vanishes. In this case we need to understand, whether the theory is irretrievably sick beyond perturbation theory, or can be salvaged. We start to suspect that far from sickness this is a sign of health: we have tried to “integrate out” a “massless particle”—it shouldn't be surprising that we run into trouble, since we're moving along flat directions in field space.

These are the zero dimensional avatars of the zero modes of the Dirac operator in field theory. So, following standard procedure, we must omit the point $x = -m^2/\lambda$, when we evaluate the Jacobian (i.e. when we evaluate the fermionic determinant) and “integrate over

the zeromode". Here this amounts to "excising" the point $x = -m^2/\lambda$, i.e. evaluating the following partition function

$$Z_\varepsilon = \int_{-\infty}^{-\frac{m^2}{\lambda}-\varepsilon} dx \underbrace{\text{sign}(m^2 + \lambda x)}_{=-1} (m^2 + \lambda x) e^{-\frac{1}{2}(m^2 x + \frac{\lambda}{2} x^2)^2} + \int_{-\frac{m^2}{\lambda}+\varepsilon}^{\infty} dx \underbrace{\text{sign}(m^2 + \lambda x)}_{=+1} (m^2 + \lambda x) e^{-\frac{1}{2}(m^2 x + \frac{\lambda}{2} x^2)^2} \quad (16)$$

and using it to compute the moments $\langle x^p \rangle$. Since we are not integrating over the entire real line—we have deleted the interval $[-m^2/\lambda - \varepsilon, -m^2/\lambda + \varepsilon]$ —we have broken supersymmetry explicitly [1, 3]. However we are interested in the limit, $\varepsilon \rightarrow 0$ and would like to understand, whether, in this limit, supersymmetry is recovered, or not.

The answer, in fact, is negative—and it isn't hard to see why. The reason is that the zero of the Jacobian is not, also, a zero of the action. Were this the case, then supersymmetry would be recovered. The proof of this statement goes as follows:

In eq. (16) the Jacobian no longer vanishes in each integral. Therefore we can perform the change of variables, $F = m^2 x + (\lambda/2)x^2$, in each and obtain the expression

$$Z_\varepsilon = - \int_{-\frac{m^4}{2\lambda} + \frac{\varepsilon^2 \lambda}{2}}^{-\frac{m^4}{2\lambda} + \frac{\varepsilon^2 \lambda}{2}} dF e^{-F^2/2} + \int_{-\frac{m^4}{2\lambda} + \frac{\varepsilon^2 \lambda}{2}}^{\infty} dF e^{-F^2/2} = 2 \int_{-\frac{m^4}{2\lambda} + \frac{\varepsilon^2 \lambda}{2}}^{\infty} dF e^{-F^2/2} \quad (17)$$

This is the exact, regularized, partition function for the auxiliary field.

For ε finite, even if we take the limit $\varepsilon \rightarrow 0$, with m and λ fixed, the lower limit of integration tends to a finite value, $-m^4/(2\lambda)$. This signals supersymmetry breaking, in the limit $\varepsilon \rightarrow 0$. The reason is that, as $\varepsilon \rightarrow 0$, the curve $F = m^2 x + (\lambda/2)x^2$ stays fixed—and the minimum of the right hand side is not equal to zero.

This, however, is easily arranged. It suffices to perform the following change of variables:

$$F = c + m^2 x + (\lambda/2)x^2 = c - \frac{m^4}{2\lambda} + \frac{\lambda}{2} \left(x + \frac{m^2}{\lambda} \right)^2 \quad (18)$$

and fix the constant c by the condition that, in the limit $\varepsilon \rightarrow 0$, the lower bound of the range of integration is at zero. In this case we immediately find that $c = m^4/(2\lambda)$. For this value of the coefficient of the linear term, F becomes a perfect square, whose (double) zero coincides with the (simple) zero of the Jacobian.

It would seem that supersymmetry has been thereby restored. Alas, this is not the case: having rendered the auxiliary field a perfect square means that its range is not the whole real axis, but, only, the non-negative part of it. Therefore the integration domain has a boundary—and that leads to supersymmetry breaking:

$$\langle F \rangle = \frac{2 \int_0^\infty dF e^{-F^2/2} F}{2 \int_0^\infty dF e^{-F^2/2}} \neq 0 \quad (19)$$

We realize that the fundamental reason is that the curve, $F(x)$, eq. (18), has a global maximum–or minimum, depending on the sign of λ –that prevents us from extending the integration range to the full real axis. We would need it to have, at most, an inflection point. An example of this is provided by the quartic superpotential,

$$U(x) = \frac{m^2}{2}x^2 + \frac{\lambda}{4!}x^4 + cx \Leftrightarrow F = \frac{dU}{dx} = m^2x + \frac{\lambda}{6}x^3 + c \quad (20)$$

If $m^2 > 0, \lambda > 0$ then the Jacobian, $U''(x) = m^2 + (\lambda/2)x^2$ is positive definite, therefore the partition function becomes a Gaussian integral, centered at zero and whose integration range is the whole real axis. Supersymmetry is realized in the Wigner mode.

If $m^2 < 0, \lambda > 0$, then the Jacobian vanishes at two points, $x = \pm\sqrt{-2m^2/\lambda}$. Its sign is positive outside of the interval $(-\sqrt{-2m^2/\lambda}, \sqrt{-2m^2/\lambda})$ and negative within. Therefore, the partition function takes the following form

$$Z_\varepsilon = \int_{-\infty}^{-\sqrt{-2m^2/\lambda}-\varepsilon} dx \left(m^2 + \frac{\lambda}{2}x^2 \right) e^{-\frac{1}{2}F^2} - \int_{-\sqrt{-2m^2/\lambda}+\varepsilon}^{\sqrt{-2m^2/\lambda}-\varepsilon} dx \left(m^2 + \frac{\lambda}{2}x^2 \right) e^{-\frac{1}{2}F^2} + \int_{\sqrt{-2m^2/\lambda}+\varepsilon}^{\infty} dx \left(m^2 + \frac{\lambda}{2}x^2 \right) e^{-\frac{1}{2}F^2} \quad (21)$$

We may perform the change of variables, $F = c + m^2x + (\lambda/6)x^3$, in each integral. The partition function becomes

$$Z_\varepsilon = \int_{-\infty}^{F(-\sqrt{-2m^2/\lambda}-\varepsilon)} dF e^{-F^2/2} - \int_{F(-\sqrt{-2m^2/\lambda}+\varepsilon)}^{F(\sqrt{-2m^2/\lambda}-\varepsilon)} dF e^{-F^2/2} + \int_{F(\sqrt{-2m^2/\lambda}+\varepsilon)}^{\infty} dF e^{-F^2/2} \quad (22)$$

and we notice that, since $F(x)$ is decreasing in the middle integral, $F(-\sqrt{-2m^2/\lambda}) > F(\sqrt{-2m^2/\lambda})$. Therefore, in the limit $\varepsilon \rightarrow 0$, which is smooth, the contribution of the middle integral cancels out the contribution from the overlap between the two other integrals, we are left with an integral over the whole real axis–and supersymmetry is restored. The linear term doesn't play any role at all, as far as realization or not of supersymmetry is concerned–it can only affect how supersymmetry is broken: For the cubic superpotential, even if its coefficient doesn't take the special value, $m^4/(2\lambda)$, the range of integration for F still cannot cover the whole real axis and supersymmetry will be broken. For the quartic superpotential the linear term is completely invisible, as far as supersymmetry breaking is concerned.

This analysis settles the issue of supersymmetry breaking, in principle, for these models. We would like, however, to understand the consequences for the moments of the scalar.

In the remainder of this section we shall focus on the cubic superpotential and establish how the relations between the moments of the auxiliary field, F , even when supersymmetry

is broken in the way we have seen, imply, nonetheless, relations between the moments of the scalar.

We start from the partition function of the auxiliary field itself, since it is now in a well-defined form and deduce recursion relations between its moments. These are simple enough, that we may solve them exactly. However they are relations between moments of the auxiliary field—to render them effective, we need the moments of the scalar.

In the following subsection we establish well-defined expressions for the moments of the scalar variable, x , itself and use an exact inversion of the “Nicolai map”, $F = dU/dx$, to express the moments of x in terms of the moments of F . The identities for the moments of the auxiliary field thus become identities for the moments of the scalar. However these identities do not carry additional information, since they simply express the Nicolai map, which is exact.

Such information is obtained from our calculation in the next subsection 3.3. There we compute the moments of the scalar, without recourse to the auxiliary field at all. We obtain in this way expressions that involve moments with respect to an, *a priori*, completely different distribution. Nevertheless, the way we conducted the transformations implies that the moments thus obtained must, when substituted in the expression for the auxiliary field, lead to the identities that it satisfies. In our example we can see explicitly that we obtain the same expression as that from inverting the Nicolai map. In more complicated cases, however, this, second, approach is the only one available. The simplest example is that of the quartic superpotential. We check our expression for the moments by substituting them in the identities satisfied by the auxiliary field. It is here that we really needed to compute the integrals numerically. We compute several identities, namely $\langle F \rangle = 0$ while varying the couplings and verify that supersymmetry is, indeed, realized, to numerical accuracy.

The true payoff, of course, lies in the moments for the scalar, for whose distribution we do not have an explicit expression, but which we can, in principle, reconstruct. We can compute, in particular, for the cubic superpotential, the exact expression for the connected fourth moment, (that, in the field theory context, would control the scattering of two scalars to two scalars) and show that, for the cubic superpotential, it never vanishes. Therefore the distribution for the scalar is always non-Gaussian.

3.1 The moments of the auxiliary field

From the expression of the partition function for the auxiliary field, eq. (17) we deduce the following recursion relation for the moments

$$\begin{aligned} \frac{1}{Z_\varepsilon} 2 \int_{c - \frac{m^4}{2\lambda} + \frac{\varepsilon^2 \lambda}{2}}^{\infty} dF \frac{d}{dF} \left(F^k e^{-F^2/2} \right) &= - \underbrace{\left(c - \frac{m^4}{2\lambda} + \frac{\varepsilon^2 \lambda}{2} \right)^k \frac{2}{Z_\varepsilon} e^{-\frac{1}{2} \left(c - \frac{m^4}{2\lambda} + \frac{\varepsilon^2 \lambda}{2} \right)^2}}_{B_k} \\ \frac{1}{Z_\varepsilon} 2 \int_{c - \frac{m^4}{2\lambda} + \frac{\varepsilon^2 \lambda}{2}}^{\infty} dF \frac{d}{dF} \left(F^k e^{-F^2/2} \right) &= k \langle F^{k-1} \rangle_F - \langle F^{k+1} \rangle_F = B_k \end{aligned} \quad (23)$$

whence we deduce that the odd moments and the even moments decouple:

$$\begin{aligned}\langle F^{2l+1} \rangle_F &= 2l \langle F^{2l-1} \rangle_F - B_{2l} \\ \langle F^{2l} \rangle_F &= (2l-1) \langle F^{2(l-1)} \rangle_F - B_{2l-1}\end{aligned}\tag{24}$$

Since Z_ε exists, $\langle 1 \rangle_F = 1$, $\langle F \rangle_F$ and $\langle F^2 \rangle_F$ exist as well, all higher moments of the auxiliary field are uniquely specified by the first and second moments.

These identities do not constrain the 1-point function, $\langle F \rangle_F$, whose expression is

$$\langle F \rangle_F = \frac{1}{Z_{\varepsilon \rightarrow 0}} 2 \int_{c - \frac{m^4}{2\lambda}}^{\infty} dF e^{-F^2/2} F = \frac{e^{-\frac{1}{2}(c - \frac{m^4}{2\lambda})^2}}{2 \int_{c - \frac{m^4}{2\lambda}}^{\infty} dF e^{-F^2/2}}\tag{25}$$

This result is interesting for several reasons: (a) A consequence of the “classical equation of motion”, $F = U'(x)$, for the auxiliary field, was that $F = \eta$, thus $\langle F \rangle = 0$. This equation is, obviously, in contradiction with eq. (25). The resolution is that the symmetry is spontaneously broken by the interaction with the noise (parametrized by the fermions). (b) If we draw the distribution,

$$\rho(F) = \frac{e^{-F^2/2}}{\int_{c - \frac{m^4}{2\lambda}}^{\infty} dF e^{-F^2/2}}\tag{26}$$

for different values of the lower limit, $c - (m^4/(2\lambda))$, cf. fig. 1, we notice that its maximum is at $F^* = 0$, which is different from $\langle F \rangle_F$. Thus, $\rho(\langle F \rangle) < \rho(F^*)$. This means that, even though it is $\langle F \rangle$ that controls, whether supersymmetry is broken, or not, $F^* = 0$ is the “typical” value that will be drawn from $\rho(F)$. The ratio,

$$\frac{\rho(\langle F \rangle)}{\rho(F^*)} = \exp \left[-\frac{1}{2} \left(\frac{e^{-\frac{1}{2}(c - \frac{m^4}{2\lambda})^2}}{2 \int_{c - \frac{m^4}{2\lambda}}^{\infty} dF e^{-F^2/2}} \right)^2 \right]\tag{27}$$

We plot this ratio, as a function of the control parameter, $c - (m^4/(2\lambda))$, in fig. 2. The result is that, for $c - m^4/(2\lambda) \approx -0.3$, the ratio falls to ~ 0.75 . For $|c - (m^4/(2\lambda))| > 1$, this ratio is, practically, equal to one. But, for these values of the control parameter, $\rho(F)$ is, also, very small, so large samples are required.

Therefore, if $c = m^4/(2\lambda)$, in which case $B_k = 0$, we would be tempted to conclude that supersymmetry is realized, even though it is spontaneously broken, if we couldn’t distinguish the typical from the average value. On the other hand, this might, also, imply that supersymmetry breaking effects are much harder to detect than expected, due to this background. This issue certainly deserves further study in more realistic cases. In disordered systems this was noted in ref. [14]; in ref. [15] it was shown that it is, indeed, possible to distinguish the typical from the average value of certain correlators in disordered spin chains. If $c \neq m^4/(2\lambda)$, then $B_k \neq 0$ and the identities satisfied by the moments of the auxiliary field acquire “anomalous” terms.

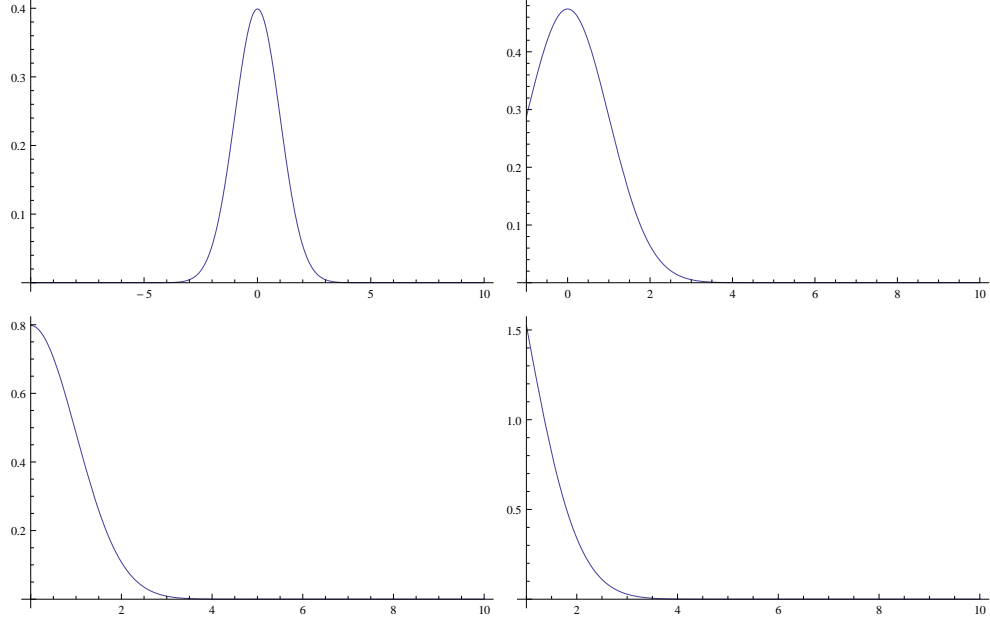


Figure 1: Some examples for the distribution of the auxiliary field, $\rho(F)$, vs. $F \geq (c - (m^4/(2\lambda)))$, for different values of $c - (m^4/(2\lambda))$. (a) $c - m^4/(2\lambda) = -10$ —here $F^* = 0 \approx \langle F \rangle$; (b) $c - (m^4/(2\lambda)) = -1$ —here $F^* = 0 \neq \langle F \rangle$, F^* is the typical value. (c) $c = m^4/(2\lambda)$ —here, too, $F^* = 0 \neq \langle F \rangle$. (d) $c - (m^4/(2\lambda)) = 1$ —here $F^* = 0$ doesn't lie in the sampling domain, $F^* \neq \langle F \rangle \neq 0$.

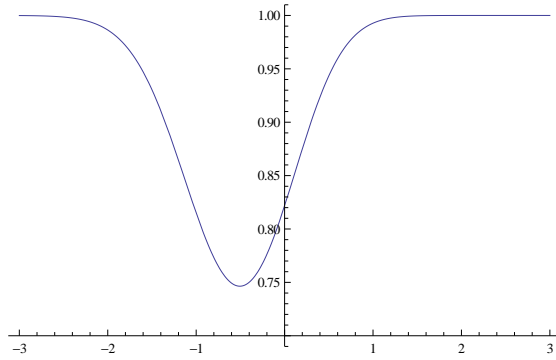


Figure 2: $\rho(\langle F \rangle)/\rho(F^*)$ as a function of $c - (m^4/(2\lambda))$.

Nevertheless, from the relation between the auxiliary field and the scalar, eqs. (24) lead to constraints on the moments of the scalar.

We shall present two approaches for computing its moments, $\langle x^p \rangle$, none of which entails an expansion in any coupling constant, but is exact, (to machine precision). The first simply relies on the fact that we can solve quadratic algebraic equations explicitly; the second dispenses with this requirement, but relies on the ability of evaluating integrals numerically.

3.2 Inverting the Nicolai map

The relations obtained above are quite model independent—the existence of the boundary limit of integration, B , doesn't depend on a specific model, only on a class as a whole—to give them content we must replace the auxiliary field by the derivative of the superpotential under study and define how we compute the moments of the scalar.

In this subsection we shall assume that we can solve the equation of motion of the auxiliary field,

$$F = \frac{dU}{dx}$$

exactly. This is what is meant by “inverting the Nicolai map”. We shall use this knowledge to write the regularized moments, $\langle x^p \rangle_\varepsilon$ of the scalar in a form that is particularly suited towards numerical calculation and shall try to check, whether these moments, when inserted in the expression $\langle (dU/dx)^q \rangle_\varepsilon$, in the limit $\varepsilon \rightarrow 0$, lead to identities consistent with Wick's theorem.

Since we are particularly interested in the role of the linear term of the superpotential, we shall include it, with an arbitrary coefficient.

The expression for the regularized moments of the scalar variable, x reads as follows:

$$\langle x^p \rangle_\varepsilon = \frac{-\int_{-\infty}^{-\frac{m^2}{\lambda}-\varepsilon} dx (m^2 + \lambda x) x^p e^{-\frac{1}{2}(c+m^2x+\frac{\lambda}{2}x^2)^2} + \int_{-\frac{m^2}{\lambda}+\varepsilon}^{\infty} dx (m^2 + \lambda x) x^p e^{-\frac{1}{2}(c+m^2x+\frac{\lambda}{2}x^2)^2}}{-\int_{-\infty}^{-\frac{m^2}{\lambda}-\varepsilon} dx (m^2 + \lambda x) e^{-\frac{1}{2}(c+m^2x+\frac{\lambda}{2}x^2)^2} + \int_{-\frac{m^2}{\lambda}+\varepsilon}^{\infty} dx (m^2 + \lambda x) e^{-\frac{1}{2}(c+m^2x+\frac{\lambda}{2}x^2)^2}} \quad (28)$$

We perform the change of variables,

$$F = c + m^2x + \frac{\lambda}{2}x^2 \Leftrightarrow dF = dx(m^2 + \lambda x)$$

whereupon the moments are computed as

$$\langle x^p \rangle_\varepsilon = \frac{\int_{c-\frac{m^4}{2\lambda}+\frac{\varepsilon^2\lambda}{2}}^{\infty} dF e^{-F^2/2} [x_-(F)^p + x_+(F)^p]}{2 \int_{c-\frac{m^4}{2\lambda}+\frac{\varepsilon^2\lambda}{2}}^{\infty} dF e^{-F^2/2}} \quad (29)$$

where

$$x_{\pm}(F) = -\frac{m^2}{\lambda} \pm \sqrt{\frac{2}{\lambda} \left[F - \left(c - \frac{m^4}{2\lambda} \right) \right]} \quad (30)$$

We end up with the following compact expression for the regularized moments, $\langle x^p \rangle_\varepsilon$:

$$\begin{aligned} \langle x^p \rangle_\varepsilon &= \left(-\frac{m^2}{\lambda}\right)^p 2 \sum_{k=0, k \text{ even}}^p \binom{p}{k} \frac{\int_{c-\frac{m^4}{2\lambda}+\frac{\varepsilon^2\lambda}{2}}^\infty dF e^{-F^2/2} \left(\frac{2\lambda}{m^4} \left(F - c + \frac{m^4}{2\lambda}\right)\right)^{k/2}}{2 \int_{c-\frac{m^4}{2\lambda}+\frac{\varepsilon^2\lambda}{2}}^\infty dF e^{-F^2/2}} = \\ &\left(\frac{m^2}{\lambda}\right)^p (-)^p \sum_{k=0}^p \binom{p}{2k} \left(\frac{2\lambda}{m^4}\right)^k \langle F^k \rangle_F \end{aligned} \quad (31)$$

where $\langle F^k \rangle_F$ are computed using the distribution (26).

From these expressions we can, in principle, recover the density, $\rho(x)$ and the action, $S(x)$.

It must be stressed that $\langle x^l \rangle$ are the *exact* moments. They are the coefficients in the Taylor series expansion of the Fourier transform, $\tilde{\rho}(k)$, of the density,

$$\rho(x) = \frac{e^{-S(x)}}{\int_{-\infty}^{\infty} du e^{-S(u)}}$$

around $k = 0$:

$$\begin{aligned} \tilde{\rho}(k) &= \int_{-\infty}^{\infty} dx e^{ikx} \rho(x) \Leftrightarrow i^{-l} \frac{d^l \tilde{\rho}(k)}{dk^l} \Big|_{k=0} = \int_{-\infty}^{\infty} dx x^l \rho(x) = \langle x^l \rangle \Rightarrow \\ \tilde{\rho}(k) &= \sum_{l=0}^{\infty} \frac{(ik)^l}{l!} \langle x^l \rangle \Rightarrow \rho(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} \tilde{\rho}(k) \end{aligned} \quad (32)$$

While the (exact) moments in eq. (31) are certainly well defined quantities, whether they do, indeed, define a density, $\rho(x)$ and, thus, an “action”, $S(x) = -\ln \rho(x)$, for all values of the “coupling”, $2\lambda/|m^2|^2$, is a question that deserves further study, that will be reported elsewhere.

In these calculations we relied on the fact that we could solve $F = dU/dx$ exactly. There are, however, situations when we cannot invert the Nicolai map as efficiently as done in this case. An example would be that of a quartic superpotential or many superfields. Then we would have to solve a cubic equation, or a system of non-linear equations-and this cannot be done as easily as for one quadratic equation. So we need to develop methods for computing the moments of the scalar that do not rely on the Nicolai map.

To test the approach we shall apply it to the case at hand, where a direct comparison is possible and to the quartic superpotential.

3.3 A new measure for the scalar’s moments

Our calculations were based on inverting the “Nicolai map”, $u = dU/dx$ —however this inversion didn’t involve any approximations at all, since we were “lucky” that we had to solve a quadratic equation, whose roots are known analytically. Had we chosen not to invert it—or could not invert it exactly—we would have to compute the moments, $\langle x^p \rangle$ and use them to compute the moments, $\langle F^q \rangle$, of the auxiliary field. Of course we could have computed the moments in a perturbative

expansion [2, 7], but this would weaken considerably the conclusions we could reach. With numerical methods available we can explore what happens beyond perturbation theory.

It is instructive to compare the two approaches in this case.

The idea is to compute the moments $\langle x^p \rangle$ in terms of the moments of

$$\rho(y) = \exp \left(-\frac{1}{2} \left(c - \frac{m^4}{2\lambda} + \frac{\lambda}{2} y^2 \right)^2 \right) \equiv e^{-S(y^2)} \quad (33)$$

Our starting point is eq. (28). We set

$$y \equiv x + \frac{m^2}{\lambda}$$

and write the numerator as follows:

$$\lambda \left(- \int_{-\infty}^{-\varepsilon} dy e^{-S(y^2)} y \left(y - \frac{m^2}{\lambda} \right)^p + \int_{\varepsilon}^{\infty} dy e^{-S(y^2)} y \left(y - \frac{m^2}{\lambda} \right)^p \right) \quad (34)$$

The denominator is obtained setting $p = 0$ in this expression. Once more, the $\varepsilon \rightarrow 0$ limit is smooth, so the moments $\langle x^p \rangle$ may be computed by

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \langle x^p \rangle_{\varepsilon} &= \lim_{\varepsilon \rightarrow 0} \frac{- \int_{-\infty}^{-\varepsilon} dy e^{-S(y^2)} y \left(y - \frac{m^2}{\lambda} \right)^p + \int_{\varepsilon}^{\infty} dy e^{-S(y^2)} y \left(y - \frac{m^2}{\lambda} \right)^p}{\int_{\varepsilon^2}^{\infty} dy e^{-S(y)}} = \\ &= \frac{\int_0^{\infty} dy e^{-\frac{1}{2} \left(c - \frac{m^4}{2\lambda} + \frac{\lambda}{2} y^2 \right)^2} y \left[(-)^p \left(y + \frac{m^2}{\lambda} \right)^p + \left(y - \frac{m^2}{\lambda} \right)^p \right]}{\int_0^{\infty} dy e^{-\frac{1}{2} \left(c - \frac{m^4}{2\lambda} + \frac{\lambda}{2} y^2 \right)^2}} \end{aligned} \quad (35)$$

The numerator can be simplified:

$$\begin{aligned} \left(-y - \frac{m^2}{\lambda} \right)^p + \left(y - \frac{m^2}{\lambda} \right)^p &= \sum_{k=0}^p \binom{p}{k} \left(-\frac{m^2}{\lambda} \right)^{p-k} [(-y)^k + y^k] = \\ &= 2 \left(-\frac{m^2}{\lambda} \right)^p \sum_{k=0}^p \binom{p}{2k} \left(\frac{\lambda}{m^2} \right)^{2k} y^{2k} \end{aligned} \quad (36)$$

This leads to the change of variables $2ydy = d(y^2)$ in the numerator. After some further straightforward algebra, we end up with the expression

$$\langle x^p \rangle = \left(\frac{m^2}{\lambda} \right)^p (-)^p \sum_{k=0}^p \binom{p}{2k} \left(\frac{2\lambda}{m^4} \right)^k \frac{\int_0^{\infty} dy e^{-\frac{1}{2} \left(c - \frac{m^4}{2\lambda} + y \right)^2} y^k}{\int_0^{\infty} dy e^{-\frac{1}{2} \left(c - \frac{m^4}{2\lambda} + y \right)^2}} \quad (37)$$

which, we perceive, after the fact, is identical to eq. (31)! We didn't need to solve the equation $F = dU/dx$ —we only used the knowledge of the root, $x = -m^2/\lambda$, of $d^2U/dx^2 = m^2 + \lambda x = 0$, around which we needed to impose the excision.

For the quartic superpotential, with $m^2 < 0, \lambda > 0$, the procedure is the following: We write $U''(x) = m^2 + (\lambda/2)x^2 = (\lambda/2)(x - x_1)(x - x_2)$, with $x_1 < x_2$. If we define the “reduced partition function”, $\zeta_{a,b}$, by the expression

$$\zeta_{a,b} \equiv \int_a^b dx e^{-\frac{1}{2}(dU/dx)^2} \quad (38)$$

the moments, $\langle x^p \rangle$, are given by the expression

$$\begin{aligned} \langle x^p \rangle &= \frac{\langle U'' x^p \rangle_{-\infty, \infty}}{\langle U'' \rangle_{-\infty, \infty}} \equiv \\ \lim_{\varepsilon \rightarrow 0} \langle x^p \rangle_\varepsilon &= \lim_{\varepsilon \rightarrow 0} \frac{\zeta_{-\infty, x_1-\varepsilon} \langle U'' x^p \rangle_{-\infty, x_1-\varepsilon} - \zeta_{x_1+\varepsilon, x_2-\varepsilon} \langle U'' x^p \rangle_{x_1+\varepsilon, x_2-\varepsilon} + \zeta_{x_2+\varepsilon, \infty} \langle U'' x^p \rangle_{x_2+\varepsilon, \infty}}{\zeta_{-\infty, x_1-\varepsilon} \langle U'' \rangle_{-\infty, x_1-\varepsilon} - \zeta_{x_1+\varepsilon, x_2-\varepsilon} \langle U'' \rangle_{x_1+\varepsilon, x_2-\varepsilon} + \zeta_{x_2+\varepsilon, \infty} \langle U'' \rangle_{x_2+\varepsilon, \infty}} \end{aligned} \quad (39)$$

where $\langle \mathcal{O} \rangle_{a,b}$ denotes average with respect to $\zeta_{a,b}$. Of course here this is just a convenient shorthand—it is more efficient to use Simpson’s rule than Monte Carlo (we’ve used both). But for the more general cases, these expressions generalize better. They are “templates”: for concrete applications it is useful to try to simplify them, in order to generate efficient code, as was, already, done, for the cubic superpotential.

In the following section we shall study the identities that these moments satisfy.

4 Moments and Identities

Let us start with the cubic superpotential.

The expressions for the moments display, first of all, scaling:

$$\langle x^p \rangle_\varepsilon = \left(\frac{m^2}{\lambda} \right)^p f_p \left(\frac{\lambda}{m^4} \right) \quad (40)$$

The prefactor, $(m^2/\lambda)^p$ indicates the “canonical dimension” of the p –th moment—which coincides with its “scaling dimension”. The “scaling function”, $f_p(\cdot)$, is a function only of the combination, λ/m^4 . The combination m^2/λ simply sets the scale, it is the combination λ/m^4 that plays the role of the coupling constant. In other words, the combinations that are relevant for describing the moments are not m^2 and λ , but m^2/λ and λ/m^4 .

The moments are perfectly regular functions of the cutoff, ε , and have a smooth limit, as it is removed, $\varepsilon \rightarrow 0$. The coefficient, c , of the linear term of the superpotential simply affects the value of integration limit—even if this can be pushed to infinity in the denominator (thanks to the factor of 2), it survives in the numerator. However, if

$$c = \frac{m^4}{2\lambda} \quad (41)$$

then the identities satisfied by the auxiliary field, F imply that supersymmetry is spontaneously broken: $\langle F \rangle_F \neq 0$ but all other identities that express the Gaussian distribution for F are exactly satisfied.

If $c \neq m^4/(2\lambda)$, then the fermion is still the goldstino, but the WT identities have anomalies.

If we compute $\langle F \rangle$, in the limit $\varepsilon \rightarrow 0$ and for $c = m^4/(2\lambda)$, we find that

$$\langle F \rangle = \frac{m^4}{2\lambda} + m^2 \langle x \rangle + \frac{\lambda}{2} \langle x^2 \rangle = \underbrace{\frac{m^4}{2\lambda} - \frac{m^4}{\lambda} + \frac{m^4}{2\lambda}}_{=0} + \underbrace{\frac{\int_0^\infty dF e^{-F^2/2} F}{2 \int_0^\infty dF e^{-F^2/2}}}_{\neq 0} \quad (42)$$

identical with the expression (25), when $c = m^4/(2\lambda)$.

Similarly, if we compute $\langle F^2 \rangle$, when $c = m^4/(2\lambda)$, in the limit $\varepsilon \rightarrow 0$, we find

$$\begin{aligned} \langle F^2 \rangle &= \left\langle \frac{\lambda^2}{4} \left(x + \frac{m^2}{\lambda} \right)^4 \right\rangle = \\ &= \frac{\lambda^2}{4} \left(\langle x^4 \rangle + 4 \langle x^3 \rangle \frac{m^2}{\lambda} + 6 \langle x^2 \rangle \left(\frac{m^2}{\lambda} \right)^2 + 4 \langle x \rangle \left(\frac{m^2}{\lambda} \right)^3 + \left(\frac{m^2}{\lambda} \right)^4 \right) \end{aligned} \quad (43)$$

Substituting the expressions for the moments (31), we find, indeed, that $\langle F^2 \rangle = \langle F^2 \rangle$, i.e. that the right hand side of eq. (43) is equal to 1.

These are but two examples of eq. (23), which implies that the moments, $\langle x^p \rangle$, of the scalar, given by eq. (31) or (35), satisfy the following identities

$$\left\langle \left(c - \frac{m^4}{2\lambda} + \frac{\lambda}{2} \left(x + \frac{m^2}{\lambda} \right)^2 \right)^{k+1} \right\rangle = k \left\langle \left(c - \frac{m^4}{2\lambda} + \frac{\lambda}{2} \left(x + \frac{m^2}{\lambda} \right)^2 \right)^{k-1} \right\rangle - B_k \quad (44)$$

where B_k is defined by eq. (23). These identities would be quite difficult to guess, if the only inputs were the moments, $\langle x^p \rangle$, given by eqs. (31) or (37).

The stochastic approach, therefore, helps us (a) to obtain concrete expressions for the moments, $\langle x^p \rangle$, of the “physical” variable, x , without prior knowledge of the probability density, $\rho(x)$, and (b) to deduce identities between these moments, that would be very difficult to deduce from scratch.

An interesting question is, whether the moments are compatible with a Gaussian distribution for the scalar. One way to address it is to compute the connected four-point function,

$$\langle x^4 \rangle_c \equiv \langle x^4 \rangle - 3 \langle x^2 \rangle^2 + 3 \langle x^2 \rangle \langle x \rangle^2 - \langle x^3 \rangle \langle x \rangle \quad (45)$$

which vanishes for a Gaussian distribution as a consequence of Wick’s theorem. Using eqs. (31), or (35), we find, indeed, that it doesn’t vanish. By dimensional analysis and scaling it is of the form

$$\langle x^4 \rangle_c = \left(\frac{m^2}{\lambda} \right)^4 f \left(\frac{\lambda}{m^4} \right) \quad (46)$$

and we expect that $f(g) \rightarrow 0$ as $g \rightarrow 0$, since, for $\lambda/m^4 \rightarrow 0$ the superpotential is quadratic. Using either of the expressions (31),(35) we find that

$$f\left(\frac{\lambda}{m^4}\right) = \frac{4(6-\pi)}{\pi} \left(\frac{2\lambda}{m^4}\right)^2 \quad (47)$$

for the case when $c = m^4/(2\lambda)$. We can, of course, compute it for any value of the coefficient of the linear term. We conclude that, for any, finite, value of λ/m^4 , the distribution of the scalar differs markedly from that of a Gaussian, and the “non-Gaussian” effects become more marked with increasing λ/m^4 at fixed m^2/λ .

Now let us turn to the quartic superpotential, when $m^2 < 0, \lambda > 0$. As we saw, despite the presence of fermionic zeromodes, supersymmetry is realized. However since we can't invert, efficiently, the Nicolai map, but compute the moments of the scalar by a way that doesn't, immediately, seem related to the auxiliary field, to show that the moments, computed this way, do satisfy the stochastic identities is not completely obvious.

The master identity, that expresses the fact that the auxiliary field, $F = m^2x + (\lambda/6)x^3$, in the limit when the cutoff is removed, is drawn from a Gaussian distribution, reads

$$\begin{aligned} k \langle F^{k-1} \rangle_F &= \langle F^{k+1} \rangle_F \Rightarrow \\ k \left\langle \left(-|m^2|x + \frac{\lambda}{6}x^3 \right)^{k-1} \right\rangle &= \left\langle \left(-|m^2|x + \frac{\lambda}{6}x^3 \right)^{k+1} \right\rangle \end{aligned} \quad (48)$$

along with the “initial condition”, $\langle F \rangle = 0$. The master identity, for $k = 1$, implies the first, non-trivial, relation

$$\langle F^2 \rangle_F = \langle 1 \rangle_F = 1 \Rightarrow \left\langle \left(-|m^2|x + \frac{\lambda}{6}x^3 \right)^2 \right\rangle = 1 \quad (49)$$

We shall start with the 1-point function for the auxiliary field: Supersymmetry implies that $\langle F \rangle_F = 0$, therefore,

$$-|m^2| \langle x \rangle + \frac{\lambda}{6} \langle x^3 \rangle = 0 \quad (50)$$

Once more it's useful to work with rescaled variables. Let us set

$$x \equiv X \left(\frac{2|m^2|}{\lambda} \right)^{1/2} \quad (51)$$

The moments then are calculated through

$$\langle x^p \rangle = \left(\frac{2|m^2|}{\lambda} \right)^{p/2} \frac{1}{Z} \int_{-\infty}^{\infty} dX (-1 + X^2) X^p e^{-\frac{|m^2|^3}{\lambda} (-X + \frac{X^3}{3})^2} \equiv \left(\frac{2|m^2|}{\lambda} \right)^{p/2} \frac{\langle (-1 + X^2) X^p \rangle}{\langle (-1 + X^2) \rangle} \quad (52)$$

The scaling relation for the moments of the quartic superpotential becomes

$$\langle x^p \rangle = \left(\frac{2|m^2|}{\lambda} \right)^{p/2} h_p \left(\frac{\lambda}{|m^2|^3} \right) \quad (53)$$

and we can write the master identity (48) in terms of the rescaled variable X , in the following way

$$\left\langle \left(-X + \frac{X^3}{3} \right)^{k+1} \right\rangle = \left(\frac{\lambda}{2|m^2|^3} \right)^2 \left\langle \left(-X + \frac{X^3}{3} \right)^{k-1} \right\rangle = k \left\langle \left(-X + \frac{X^3}{3} \right)^2 \right\rangle \left\langle \left(-X + \frac{X^3}{3} \right)^{k-1} \right\rangle \quad (54)$$

with

$$\left\langle \left(-X + \frac{X^3}{3} \right)^k \right\rangle = \frac{\int_{-\infty}^{\infty} dX (-1 + X^2) \left(-X + \frac{X^3}{3} \right)^k e^{-\frac{|m^2|^3}{\lambda} \left(-X + \frac{X^3}{3} \right)^2}}{\int_{-\infty}^{\infty} dX (-1 + X^2) e^{-\frac{|m^2|^3}{\lambda} \left(-X + \frac{X^3}{3} \right)^2}} \quad (55)$$

The relation, $\langle F \rangle_F = 0$ becomes

$$-\langle X \rangle + \frac{\langle X^3 \rangle}{3} = \left\langle -X + \frac{X^3}{3} \right\rangle = 0 \quad (56)$$

If we write it out:

$$\int_{-\infty}^{\infty} dX \left(-X + \frac{X^3}{3} \right) (-1 + X^2) e^{-\frac{|m^2|^3}{\lambda} \left(-X + \frac{X^3}{3} \right)^2} = 0 \quad (57)$$

it becomes obvious, since, in this guise, the integral exists, thanks to the exponential factor and the integrand is the product of an even function and an odd function, integrated along an interval, symmetric about the origin, therefore it vanishes. Indeed, any odd moment vanishes as a result of the symmetry, $X \leftrightarrow -X$.

Let us look at a, possibly, less trivial relation, namely $\langle F^2 \rangle_F = 1$. In terms of the $\langle X^p \rangle$, it reads

$$\left\langle \left(-X + \frac{X^3}{3} \right)^2 \right\rangle = \frac{\lambda}{2|m^2|^3} \Leftrightarrow \frac{\int_{-\infty}^{\infty} dX (-1 + X^2) \left(-X + \frac{X^3}{3} \right)^2 e^{-\frac{|m^2|^3}{\lambda} \left(-X + \frac{X^3}{3} \right)^2}}{\int_{-\infty}^{\infty} dX (-1 + X^2) e^{-\frac{|m^2|^3}{\lambda} \left(-X + \frac{X^3}{3} \right)^2}} = \frac{\lambda}{2|m^2|^3} \quad (58)$$

If we could perform the change of variables, $F \equiv \sqrt{|m^2|^3/\lambda} (-X + (X^3/3))$, it would be obvious. The analysis in the previous section confirms this expectation. Let us show that it is possible to recover this result by computing the moments according to eq. (52).

If we draw $\exp \left(-(|m^2|^3/\lambda) (-X + (X^3/3))^2 \right)$ for different values of the coupling constant, $\lambda/|m^2|^3$, cf. fig. 3 that cover the interval from “weak” ($\lambda/|m^2|^3 < 1$) to “strong” ($\lambda/|m^2|^3 > 1$)

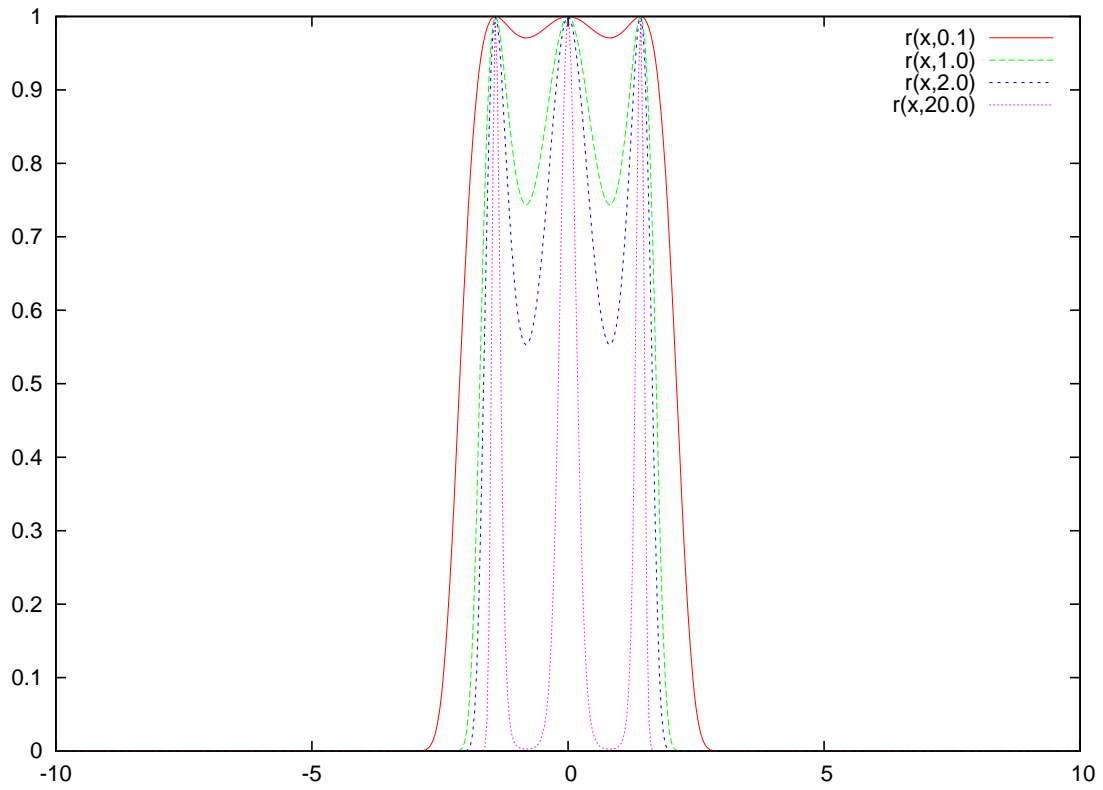


Figure 3: $\exp\left(-(|m^2|^3/\lambda)(-X + (X^3/3))^2\right)$ as a function of X for different values of $|m^2|^3/\lambda$.

coupling, we note that, for weak coupling, the density is concentrated around the “classical” minima, while, as the coupling increases, it becomes easier to explore other values, in particular around the local maxima of the “classical action”—however we do note that, for $|X| > 5$, the density becomes negligible: for $|X| \approx 5$ the density is smaller than $\sim \exp(-37^2|m^2|^3/\lambda)$. Therefore we may cutoff the integrals at $|X| = 5$.

In fig. 4 we display, for different values of $\lambda/|m^2|^3$, the absolute value of the difference

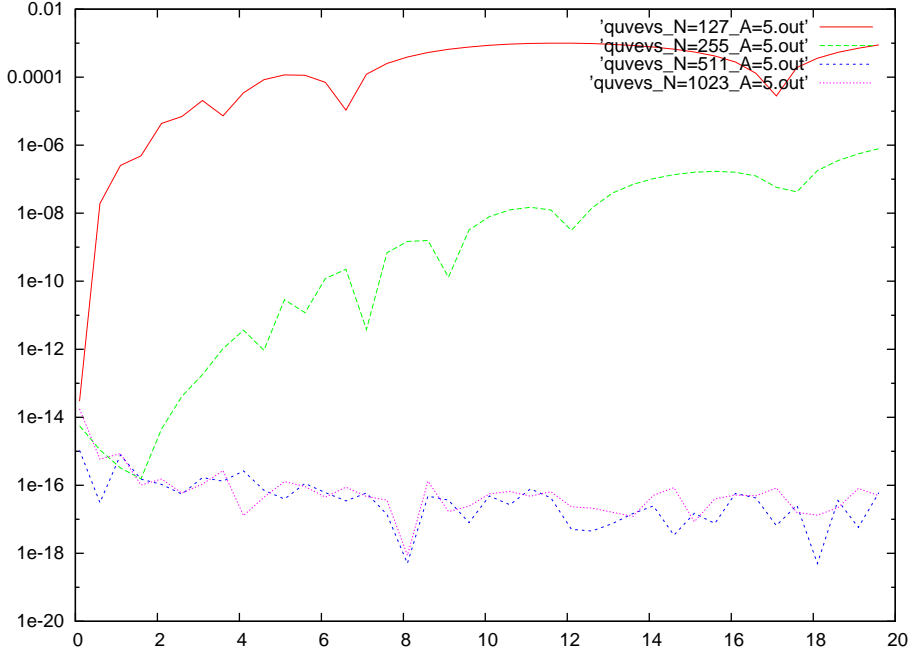


Figure 4: Absolute value of the difference of the two sides of relation (58) as a function of $|m^2|^3/\lambda$. The vertical scale is logarithmic. The different curves correspond to different values of points for Simpson’s rule, used for evaluating the integrals, in the interval $-5 \leq X \leq 5$.

between the left hand side and the right hand side of eq. (58). We find, once more, a result that is consistent with zero, to machine precision, when numerical integration errors become negligible. Our code passes the test.

5 Conclusions and Perspectives

We have studied the partition function of probability distributions of one variable, that are generated by solutions of Langevin's equation. This defines a zero-dimensional field theory. We have shown that the induced probability distribution enjoys worldpoint supersymmetry, that can be broken by surface terms, induced by fermionic zero-modes. The “fermions” enter through the noise term.

While a linear term in the superpotential, also, contributes surface terms, in fact it cannot cancel their contribution. In this way we provide concrete examples for the boundary terms studied by Parisi and Sourlas [3].

Nevertheless, this breaking doesn't imply that supersymmetry isn't useful: as we have shown, even broken, it, still imposes relations on the moments and all higher moments of the auxiliary field are determined by its first and second moments. So the auxiliary field is still drawn from a Gaussian distribution, “deformed” in an interesting way, due to the interactions.

For practical purposes, we can find examples, where supersymmetry is spontaneously broken, but the goldstino is “hard to find”, since the average value of the auxiliary field is not the “typical” value, that is drawn from its distribution.

The scalar field, on the other hand, isn't drawn from a Gaussian distribution—but we can compute its moments, at any value of the coupling, quite explicitly and they determine its distribution (its effective action, in the field theory context), in principle. We have determined exact expressions for the moments of the cubic superpotential by two independent methods: inverting the Nicolai map and by establishing the relation to the moments of the scalar part of the classical action. It is this second approach that most naturally generalizes to higher dimensions, where numerical simulations are the natural tool, which we tested on the quartic superpotential, where supersymmetry is realized, whatever the sign of the quadratic term of the superpotential.

So the situation is by no means “all or nothing”: supersymmetry either is realized and constrains everything or is broken and is not useful, but is much more interesting. The information that has not been appreciated lies in the identities between moments or correlation functions. While in the “traditional” approach to supersymmetric field theory, one starts from the “physical” (albeit classical) fields and the auxiliary fields are, indeed, quite hard to find [7], in the stochastic approach the auxiliary fields are the starting point and the “physical” fields are “emergent” quantities. In this approach supersymmetry is a natural symmetry. So to understand it better, it might seem that the stochastic approach, through the access it provides to the moments/correlation functions, might prove much more useful.

In fact a natural direction of further study is the following: The reason why the auxiliary field is drawn from a Gaussian distribution is that it was, initially, at least, identified with the noise in the Langevin equation. So we might consider other stochastic equations, where the noise is drawn from other distributions, e.g. “colored” noise [12] and, of course we can imagine further generalizations. These might lead to more general supersymmetry-breaking patterns, that definitely deserve to be studied in higher dimensions.

The most straightforward generalization is to “worldline supersymmetry”, i.e. supersymmetric quantum mechanics [9, 10, 11]. This has been, mainly, studied in a way that stresses its similarity to (non-relativistic) quantum mechanics rather than quantum field theory (an exception is ref. [10], where, however, the focus was on topology and not on the stochastic identities themselves, that were studied in ref. [7] in an approach that combined canonical and path integral approaches). Lattice studies [17] use the Nicolai map but, apparently, do not consider the stochastic identities themselves. These are “broken” by the lattice since the auxiliary field has a local propagator, that becomes ultra-local only in the continuum limit (at least for the free theory).

In this paper we have tried, on the contrary, to set up the formalism that most readily generalizes to quantum field theory, namely the (zero-dimensional analog of the) path integral formalism and the WT identities between moments. We have thereby elucidated the role of the distribution of the auxiliary field in characterizing supersymmetry breaking and developed the tools that will be useful for the higher dimensional cases.

Our results can be considered as describing the degrees of freedom that live on the boundary of Euclidean time, with supersymmetric quantum mechanics describing the “bulk” theory. For the cubic superpotential, supersymmetry is therefore broken both in the bulk [3, 5, 6, 9, 4] and on the boundaries. For the quartic superpotential it is broken only in the bulk. These statements, however, do not take into account the coupling between bulk and boundaries, which is the issue that remains to be studied (work in this direction, in the canonical formalism, was done in ref. [11]).

We have studied the case of one chiral multiplet and a natural generalization is to $N_f > 1$ chiral multiplets—which also lead naturally to theories with extended supersymmetry. In that case the technical difficulty stems from the fact that we must “excise” not just a point but a vector space, generated by the zeromode(s) of the Jacobian, $N_f \times N_f$, matrix. The use of collective coordinates [16] is the natural tool, already at this level.

Finally, another topic concerns supersymmetric gauge theories. Here the problem is how to handle gauge invariance. In stochastic quantization [18] gauge fixing is not mandatory, but only gauge invariant quantities are well-defined when the equilibrium limit is taken. If we work on the lattice, with gauge links that take values in a compact group, all observables are gauge invariant, but the gauge links are constrained: e.g. they are unitary matrices, for unitary groups. We may solve these constraints, however [19], so it will be interesting to study the corresponding Langevin equation and whether we may obtain supersymmetric gauge theories this way.

For, indeed, the question remains, how “target space” supersymmetry can be obtained by the stochastic approach. Parisi and Sourlas did succeed in obtaining the $\mathcal{N} = 2$ Wess-Zumino model in two spacetime dimensions, but found an obstruction in four dimensions. de Alfaro, Fubini, Furlan and Veneziano [7] obtained some, partial, results, in particular for gauge theories, but the issue remains open, how to write a lattice theory in this formulation. This is one reason why the “orbifold” approach [20] has since been used. It is, therefore, of interest to test its assumptions by another formulation.

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